Test 2 Numerical Mathematics 2
January 12, 2023

Duration: 90 minutes.

In front of the questions one finds the points. The sum of the points plus 1 gives the end mark of this test. Use of a calculator is allowed.

1. Consider the matrix
$$A = \begin{bmatrix} -1 & 3 & 0 & 0 \\ -1 & -2 & -2 & 0 \\ 0 & 4 & -3 & 6 \\ 0 & 0 & -2 & -4 \end{bmatrix}.$$
   (a) [0.7] Show that A is irreducible.

   (b) [1.5] Write A as the sum of a symmetric and a skew-symmetric matrix. Consider the symmetric part and localize its eigenvalues by the Gershgorin theorems. And similar for the skew-symmetric part of A

   (c) [0.8] According to Bendixson's theorem, where are the eigenvalues of A located in the complex plane based on the results in the previous part? Is A non-singular?

2. Let A be a square matrix with a complete set of eigenvectors $v_1, \cdots, v_n$ with $Av_i = \lambda_i v_i$, $|\lambda_1| > |\lambda_2| > |\lambda_3| \geq \cdots \geq |\lambda_n|$. Furthermore, let $x_1 = \sum_{i=1}^n v_i$, $x_2 = \sum_{i=1}^n \frac{1}{i} v_i$. Define $X = [x_1, x_2]$.

   (a) [1.5] Show that $Y^{(n)} = A^n X$ converges for large n to $[\lambda_1^n v_1 + \lambda_2^n v_2, \lambda_1^n v_1 + \frac{1}{2}\lambda_2^n v_2]$.

   (b) [1.5] Show that the eigenvalues θ_1 and θ_2 of the 2-dimensional (generalized) eigenvalue problem
$$(Y^{(n)})^T A Y^{(n)} \hat{y} = \theta (Y^{(n)})^T Y^{(n)} \hat{y} \tag{1}$$
   are λ_1 and λ_2 when the convergence as indicated in part a has set in. *Hint:* consider $\hat{y} = [1, -2]^T$ and $\hat{y} = [1, -1]^T$, respectively.

   (c) [1.0] Though analytically the above will give us the two dominant eigenvalues of A it will fail on a computer. Explain why this is happening. How can one remedy it?

3. Let $K^m(A, v)$ denote the Krylov subspace.

   (a) [0.5] How is the Krylov subspace $K^m(A, v)$ defined?

   (b) [1.5] Suppose $A \in \mathbb{R}^{n \times n}$ has $k < n$ different eigenvalues and A has a complete set of eigenvectors, i.e. the eigenvectors are a basis for $\mathbb{R}^n$. Show that the dimension of $K^m(A, v)$ is at most k.